The Formation of Partnerships in Social Networks

Francis Bloch and Bhaskar Dutta

Université Paris 1 and PSE, University of Warwick

Paris, September 7, 2015

Motivation

- Agents use their social network to exchange favors, information, to work on common projects, to court, etc..
- Interaction (favor exchange, common work) often arises in *dyads*: pairs of agents
- In this paper, we propose a model where agents explore their networks to form partnerships (dyads)
- Main question: **Which networks support efficient formation of patnerships – ie the maximum number of pairs**

Possible applications

- Formation of couples
- Formation of pairs to work together on homeworks and projects at school
- Formation of pairs to provide informal insurance in developing countries
- Formation of pairs pairs to exchange favors, like mowing the lawn, watching the house.. in the neighborhood
- *In this paper we focus on reciprocity in pairs and not the formation of larger groups*

Main results

- If the cost of providing favors is positive, *any network supports the efficient formation of partnerships*
- If the cost of providing favors is negative, *the only networks supporting the efficient formation of partnerships are completely elementary networks*
- When the social network admits a perfect matching, the only completely elementary networks are *complete and complete bipartite networks*.

Related literature

- Informal insurance in social networks: Bramoullé Kranton (2007), Bloch, Genicot, Ray (2008)
- Exchange of favors with social collateral: Karlan, Mobius, Rosenblat, Szeidl (2009), Ambrus, Mobius, Szeidl (2007)
- Exchange of favors in social networks with renegotiation: Jackson, Rodriguez Barraquer and Tan (2012)
- Bargaining in networks: Manea (2011), Abreu and Manea (2010) (2012)

Partnerships and networks

- Agents belong to an initial social network g
- This network evolves over time as links will be destroyed
- Time is discrete and runs as $t = 1, 2...$
- At each time t, one agent is picked at random with probability $\frac{1}{n}$. This agent will *need a favor* and look for a partnership.
- A partnership is a reciprocal agreement to exchange favors.

Values and costs

- Receiving a favor brings a flow payoff v (identical across links)
- Providing a favor entails a cost c (identical across links)
- Hence agents are perfect substitutes in the provision of favors. Favors are standardized. (different from Jackson et al and Mobius)
- All agents have the same discount factor δ
- The value of a partnership is

$$V = \frac{v - c}{n(1 - \delta)}$$

Process of partnership formation

- If agent i needs a favor and is not yet in a partnership, he will ask all his (current) neighbors in the network.
- The sequence in which neighbors are asked is random
- Each neighbor responds by Yes or No.
  - If neighbor j accepts, j pays c and the partnership is formed. Agents i and j (and all their links) leave the network
  - If neighbor j rejects, the link ij is deleted and agent i asks the next neighbor in the sequence.

Strategies equilibrium efficiency

- We look at Markov strategies
- A Markov strategy determines for each agent a response Yes or No as a function of the current network and the identity of the proposer
- A Markov equilibrium is a collection of Markov strategies such that each agent chooses his optimal response after each history.
- A Markov equilibrium is (strongly) efficient if it maximizes the sum of utilities of all the agents.
- A network g supports efficient equilibria, if, for any realization of needs and of the sequence of neighbors, all Markov equilibria of the game are efficient.

An example: the line L_4

- Consider the line 12,23,34
- Efficiency requires that the two matches 12 and 34 are formed.
- If 2 has a need and goes to 3, 3 rejects because he knows that he can form a partnership with 4.
- Hence the partnership 23 is never formed in equilibrium, and if δ is sufficiently high, the two partnerships 12, 34 are formed.
- We show that this is a general phenomenon: all networks support efficient equilibria, and the maximum number of partnerships is always formed in equilibrium.

Matchings in a graph

- Given a graph g, a matching M is a collection of edges $ij \in g$ such that no pair of edges in M have a common vertex.
- This corresponds to the intuitive notion of decomposing the graph into disjoint pairs of connected agents ij.
- A node i is *covered* in matching M if i belongs to some edge in M. Otherwise it is *exposed*
- A matching M is *maximal* in g if there is no other matching M' of g such that $M \subset M'$
- A matching M is *maximum* in g if M is maximal and there is no other maximal matching M' of g such that $|M'| > |M|$.

Matching number, perfect matchings

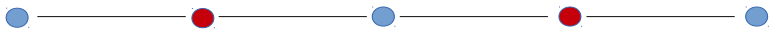
- For any graph g, we let $\mu(g)$ denote the matching number of g, ie the size of the maximum matching.
- If $\mu(g) = n/2$, the maximum matching is called a *perfect matching*
- A graph is called *perfect* if it admits a perfect matching (e.g. even complete, even line, even cycle)

Essential nodes

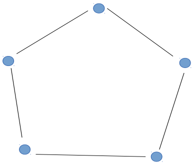
- A node i is *essential* in g if i belongs to *all maximum matchings of* g.
- Examples: Hub of a star, any node in a perfect graph..

Essential nodes: examples

In the line L_4 all nodes are essential



In the line L_5 two nodes are essential, three are inessential



In the cycle C_5 no node is essential

The Essentiality Lemma

Lemma

1. *If i is essential in g then $\mu(g) = \mu(g - i) + 1$ where $g - i$ is the graph obtained from g by deleting node i and all edges incident to i.*
2. *If i is essential in g, there exists ij in g such that j is not essential in $g - i$.*
3. *If i is not essential in g and $ij \in g$, then j is essential in $g - i$*
4. *If i is essential in g and g' is a subgraph of g such that $\mu(g) = \mu(g')$, then i is essential in g'*

The main theorem

Theorem

There exists $\overline{\delta}$ such that, for all $\delta > \overline{\delta}$, any social network g supports efficient equilibria.

Equilibrium strategies

- At any graph g, if j is essential in $g \setminus ij$, then j rejects i's request
- If j is not essential in $g \setminus ij$, then j accepts i's request.
- These strategies are the only equilibrium strategies in the game. Using the essentiality lemma, by backward induction we show:
    - If i is essential in g there always exists a neighbor j who accepts the formation of a partnership
    - If i is not essential in g, all neighbors of i reject i's request.

Evolution of the network

- In network g, if an inessential agent i is chosen, all his requests are denied and the network moves to $g \setminus i$
- In $g \setminus i$, all essential agents remain essential, some inessential agents may become essential (the set of essential agents can only grow)
- If an essential agent i is chosen, he forms a partnership with some agent j and the network moves to $g \setminus i, j$
- In $g \setminus i, j$ all essential agents remain essential, some inessential agents may become essential
- An essential agent wants to delay the formation of the partnership (in order not to pay the cost). He will delay it as long as he is sure that he will still find a partner in the network.

Sketch of the proof of the theorem

- We show that in equilibrium, the maximal number of partnerships $m(g)$ is formed.
- Suppose that the number of matches drops after the deletion of a link ij
- It must be that both i and j refused to form the link, ie remain essential in $g \setminus ij$
- But then the number of matches in the maximal matching cannot drop in $g \setminus ij$

Equilibria for low values of δ: the line

- Consider an even line of n agents
- In a line of size k, let V^k be the value of the extremal agent in the line
- We first show that V^k is increasing in k: extremal agents have a higher value if the line is longer.
- Consider any agent in the line: he accepts to form the partnership if and only if

$$\delta V - c \geq V^k,$$

  where k is the size of the line when link ij is deleted.
- The most stringent condition is obtained for agent 2, when agent 1 proposes a partnership, and the condition then is

$$\delta V - c \geq V^{n-1}.$$

Equilibria for low values of δ: the complete graph

- We consider an even complete network of size n.
- In a complete graph of size k, the value of an agent is

$$W^k = \frac{(k-1)\delta V}{n - \delta(n-k)}.$$

  This value is increasing in k so all partnerships form if and only if

$$\delta V - c \geq W^{n-1}.$$

- $W^{n-1} > V^{n-1}$, so *it is easier to support efficient equilibria in the line than in a complete network.*

Negative costs

- Suppose that $c < 0$: agents have a positive value of giving the favor.
- Then agents always accept to form a partnership
- In this game, agents rush to provide favors.

Elementary networks

- A graph g is elementary if all edges in g appear in some maximum matching.
- L_4 is not elementary but L_5 is. All cycles are elementary.

Lemma

(Lovasz and Plummer (1986)) A connected graph g is elementary if and only if, for any connected i, j *in g,* $\mu(g \setminus i, j) = \mu(g) - 1$.

Completely elementary networks

- A graph g is *completely elementary* if it is elementary and for any matching M in g, and subset of agents S matched in M, $g \setminus S$ is elementary.
- L_5 is completely elementary, C_5 is completely elementary, C_6 is elementary but not completely elementary – as it becomes the line L_4 after a pair leaves.

Characterization of networks supporting efficient equilibria

Theorem

In the model with negative costs, a social network g supports efficient equilibria if and only if it is completely elementary.

Sketch of the proof

- All partnerships are formed immediately
- The maximum number of partnerships is formed if and only if the formation of a pair i, j does not result in a disconnection of the graph
- By the characterization lemma in Lovasz Plummer, this is equivalent to the graph being completely elementary.

Bipartite graphs

- A graph is bipartite if the set of vertices can be partitioned into two subsets A and B such that agents in A are not connected and agents in B are not connected.
- A graph is complete bipartite if and only if it is bipartite and all agents in A are connected to all agents in B.

Characterization of perfect networks supporting efficient equilibria

Theorem

In the model with negative costs, a perfect social network g supports efficient equilibria if and only if it is the disjoint union of components which are either complete or complete bipartite.

Sketch of the proof

- Sufficiency: It is easy to show that complete and complete bipartite networks are completely elementary.
- Necessity:
- The proof is by induction on the size of the perfect network.
- If $n = 4$, the only networks supporting efficient equilibria are the cycle C_4 (a complete bipartite network) and the complete network K_4.
- Consider a connected network of size n. It must be that for all ij, $g \setminus i, j$ supports efficient equilibria and hence are unions of complete or complete bipartite components.
- We prove the Lemma:

A lemma on complete and complete bipartite graphs

Lemma

Consider a connected perfect graph g. If for all ij $\in$ *g,* $g \setminus i, j$ *is the disjoint union of complete or complete bipartite components,*

- $g \setminus i, j$ *must be connected*
- *g must either be complete or complete bipartite*

Remarks

- This characterization only works for perfect graphs.
- C_5 and L_5 support efficient equilibria but are not complete nor complete bipartite.

Next

- **Experiments**: Check that agents understand the game and delay the formation of partnerships as long as they can
- **Larger partnerships**: Suppose that larger partnerships also have value.
- When does a network support efficient formation of large partnerships?
- **Marriage on social networks**: Suppose that agents have different known NTU values
- When does our procedure result in efficient marriage formation?